A Note on Volatility Persistence and Structural Changes in GARCH Models

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Abstract
In this paper, we demonstrate that most of Tokyo stock return data sets have volatility persistence and it is due to a parameter change in underlying GARCH models. For testing for a parameter change, we use the cusum test, devised by Lee et al. (2003), based on the residuals from GARCH models. A simulation study shows that a parameter change in GARCH models can mislead analysts to choose an IGARCH model. We explain this phenomenon theoretically applying Hamilton (1994)’s idea.

Keywords: Structural change, GARCH(1,1) model, cusum of squares test, model misspecification.

JEL Classification Number: C14, C15, C22

1 Introduction

The problem of testing for a parameter change in statistical models has attracted much attention from many researchers: for a review of earlier works, see Csörgő and Horváth (1997). The issue became very popular in the economic time series context since economic data exhibits changes in their underlying model owing to changes in governmental policy and panic social events. For references, see Wichern, Miller and Hsu (1976), Inclán and Tiao (1994), Lee and Park (2001), and the papers cited therein. Recently, Lee et al. (2003) proposed a residual based cusum test for parameter changes in GARCH(1,1) models, and demonstrated that the cusum method is appropriate for allocating change points. The GARCH model has long been popular in modelling financial time series, and is proven to be useful in handling the data with high volatility (cf. Gouriéroux (1997)).

In actual practice, one often encounters the data with strong volatility persistence, and in this case an IGARCH model is likely to be selected as its underlying model. However, a speculation should be made on a possibility that the volatility persistence might have come from a parameter change in GARCH models: recall that this phenomenon happens in ordinary AR models as reported in Hamilton (1994, page 450). It is obvious that the parameter changes, if any, should be reflected in modelling given data set. In this paper, we demonstrate that most of Tokyo stock return data sets possess volatility persistence, and in many cases, it is a

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consequence of a parameter change in GARCH models. The details are presented in Section 2. In Section 3, we provide a simulation result showing that a parameter change in GARCH models can lead analysts to mistakenly select an IGARCH model. We explain this phenomenon theoretically borrowing Hamilton’s idea.

2 Empirical result

In this section we analyze the stock return data sets of the companies listed in Tokyo Stock Exchange, 1st section. (The number of stocks is 1068). The following GARCH(1,1) model is fitted to each data set:

\[ y_t = c + \varepsilon_t, \]
\[ \varepsilon_t = h_t \cdot \xi_t, \quad \xi_t \sim i.i.d. N(0, 1), \]
\[ h_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}^2. \]

The sums of estimated GARCH parameters are presented in Table 1.

** Table 1 here **

From Table 1, we can see that most stock data sets have volatility persistence. Note that the percentage of the cases that \( 0.95 < \alpha + \beta < 1 \) is about 80%. Now, to check whether the volatility persistence result is due to a parameter change or not, we perform a test utilizing the cusum test of Lee et al. (2003) as follows:

\[ \hat{T}_n := \frac{1}{\sqrt{n \tau}} \max_{1 \leq k \leq n} \left| \sum_{t=1}^{k} \xi_t^2 - \frac{k}{n} \sum_{t=1}^{n} \xi_t^2 \right|, \]

where

\[ \xi_t^2 = (y_t - c)^2 / h_t^2, \]
\[ \hat{\xi}_t^2 = \hat{\text{Var}}(\xi_t^2) = \frac{1}{n} \sum_{t=1}^{n} \hat{\xi}_t^4 - \left( \frac{1}{n} \sum_{t=1}^{n} \hat{\xi}_t \right)^2. \]

Lee et al. showed that \( \hat{T}_n \) converges in distribution to the sup of a standard Brownian bridge \( W^0 \): the distribution of \( \sup_{0 \leq t \leq 1} |W^0_t| \) is given in Equation (11.39) of Billingsley (1968).

Table 2 shows that about 50% of the data sets showing volatility persistence turn out to suffer from structural changes. This result strongly indicates that one needs to carefully examine a possibility of a structural change when the data shows volatility persistence.

** Table 2 here **

Here, we fit GARCH model to the return of Japan Airline data set from Jan 4, 1991 to Dec 28, 2001. According to our analysis using the \( D_k \) plot as in Inclán and Tiao (1994), only one change point was detected on Oct 1, 1997 (see the vertical line in Figure 1). The data in the first period from Jan 4, 1991 to Oct 1, 1997 appears to follow the model:

\[ y_t = -0.088 + \varepsilon_t, \]
\[ \varepsilon_t = h_t \cdot \xi_t, \]
\[ h_t^2 = 0.860 + 0.207 \varepsilon_{t-1}^2 + 0.535 h_{t-1}^2. \]
and the data in the second period follows the model

\[ y_t = -0.084 + \varepsilon_t, \]
\[ \varepsilon_t = h_t \cdot \xi_t, \]
\[ h_t^2 = 1.791 + 0.147 \varepsilon_{t-1}^2 + 0.608 h_{t-1}^2. \]

The above result shows that the GARCH parameters experience significant changes.

Meanwhile, if we ignore the change and fit the GARCH(1,1) model to the data in the whole period, the fitted model appears to be an IGARCH(1,1) model

\[ y_t = -0.065 + \varepsilon_t, \]
\[ \varepsilon_t = h_t \cdot \xi_t, \]
\[ h_t^2 = 0.224 + 0.149 \varepsilon_{t-1}^2 + 0.767 h_{t-1}^2. \]

This shows that ignoring changes can lead to a false conclusion in statistical inference.

** Figure 1 **

3 Theoretical perspective

In this section, we perform a simulation study to ensure the phenomenon described in the previous section. For this task, we consider the model

\[ y_t = 0.95 + \varepsilon_t, \quad t = 1, 2, \cdots, n, \]
\[ \varepsilon_t = h_t \cdot \xi_t, \quad \xi_t \sim \text{i.i.d.} N(0,1), \]
\[ h_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}^2. \]
\[ \alpha = 0.1 \sim 0.8, \beta = 0.1 \sim 0.8 \text{ with } \alpha + \beta \leq 0.7, \]
\[ \omega = \begin{cases} 
0.5, & t \leq \frac{n}{2}, \\
1.5, & \text{otherwise}. 
\end{cases} \]

** Table 3 here **

Table 3 shows that the estimates of \( \alpha + \beta \) are pretty much close to 1 in about 58% cases.

Now, we give a theoretical explanation for this result following Hamilton’s idea. Here, we focus on the case of a change in \( \omega \).

First, we consider the GARCH(1,1) model

\[ y_t = h_t \cdot \xi_t, \quad \xi_t \sim \text{i.i.d.} N(0,1), \]
\[ h_t^2 = \omega + \alpha y_{t-1}^2 + \beta h_{t-1}^2. \]

Set \( \nu_t = y_t^2 - h_t^2 \). Then we have

\[ \nu_t = y_t^2 - (1 - \beta B)^{-1} (\omega + \alpha y_{t-1}^2) \]

or

\[ y_t^2 = \frac{\omega}{1 - \beta} + \alpha (1 - \beta B)^{-1} y_{t-1}^2 + \nu, \]
where \{\nu_t\} can be viewed as a white noise.

Obviously, we have
\[(1 - (\alpha + \beta) B) y_t^2 = \omega + (1 - \beta B) \nu_t.\]

If \(\alpha + \beta = 1\), we have
\[
\Delta y_t^2 = \omega + \nu_t',
\]
where \(\Delta y_t^2 = (1 - B) y_t^2\), and \(\nu_t' = (1 - \beta B) \nu_t\), and \(\{\nu_t'\}\) is an MA(1) process.

Next we consider the model with a structural break at time \(n_0 < n\):
\[y_t = h_t \cdot \xi_t, \quad \xi_t \sim i.i.d. N(0, 1),\]

where
\[h_t^2 = \begin{cases} 
\omega_1 + \alpha \xi_{t-1} + \beta h_{t-1}^2, & t < n_0, \\
\omega_2 + \alpha \xi_{t-1} + \beta h_{t-1}^2, & t \geq n_0
\end{cases}\]

with \(\omega_2 \neq \omega_1\). Since
\[
y_t^2 = \begin{cases} 
\alpha (1 - \beta B)^{-1} y_{t-1}^2 + \frac{\omega_1}{1 - \beta} + \nu_t, & t < n_0, \\
\alpha (1 - \beta B)^{-1} y_{t-1}^2 + \frac{\omega_2}{1 - \beta} + \nu_t, & t \geq n_0
\end{cases}\]

we can write
\[
\Delta y_t^2 = \alpha (1 - \beta B)^{-1} \Delta y_{t-1}^2 + \eta'_t,
\]
where \(\eta'_t = \eta_t + \nu_t - \nu_{t-1}\), and
\[\eta_t = \begin{cases} 
\frac{\omega_2 - \omega_1}{1 - \beta}, & t = n_0, \\
0, & otherwise
\end{cases}\]

Applying Hamilton’s idea we may assume that
\[\eta_t = \begin{cases} 
\frac{\omega_2 - \omega_1}{1 - \beta} & \text{with probability } p, \\
0 & \text{with probability } 1 - p,
\end{cases}\]

where \(p\) is a small number to represent the idea that this is a relatively rare event. Notice that
\[E[\eta_t] = p \cdot \frac{\omega_2 - \omega_1}{1 - \beta} = c \neq 0.\]

Obviously, we have
\[(1 - \beta B) \Delta y_t^2 = (1 - \beta) c + \alpha \Delta y_{t-1}^2 + \eta''_t,\]
where
\[ \eta''_t = (1 - \beta B) (\eta'_t - c). \]

Then we have
\[ (1 - (\alpha + \beta) B) \Delta y^2_t = (1 - \beta) c + \eta''_t \]
or
\[ \Delta y^2_t = \frac{(1 - \beta) c}{1 - \alpha - \beta} + (1 - (\alpha + \beta) B)^{-1} \eta''_t \]
\[ = c' + \eta'''_t, \quad (3) \]

where \( \eta'''_t = (1 - (\alpha + \beta) B)^{-1} \eta''_t \) and \( \{\eta'''_t\} \) is an MA(\( \infty \)) process.

In view of (1) and (3), we can see that the squares of \( y_t \) in (2) can be expressed in the same fashion as those in the IGARCH(1,1) process. This indicates that GARCH(1,1) process with a break as in (2) can be described to follow an IGARCH(1,1) model as claimed earlier.

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References


| Table 1 | | | | 
|---|---|---|---|---|
| | \( \hat{\alpha} + \beta \leq 0.9 \) | \( 0.9 < \hat{\alpha} + \beta \leq 0.95 \) | \( 0.95 < \hat{\alpha} + \beta \) | total |
| Number | 57(5.3%) | 168(15.7%) | 843(79.0%) | 1068 |

| Table 2 | | | | 
|---|---|---|---|---|
| | \( \hat{\alpha} + \beta \leq 0.9 \) | \( 0.9 < \hat{\alpha} + \beta \leq 0.95 \) | \( 0.95 < \hat{\alpha} + \beta \) | total |
| Number | 22(38.6%) | 82(48.8%) | 401(48.8%) | 505(47.3%) |

| Table 3 | | | | 
|---|---|---|---|---|
| | \( \hat{\alpha} + \beta \leq 0.9 \) | \( 0.9 < \hat{\alpha} + \beta \leq 0.95 \) | \( 0.95 < \hat{\alpha} + \beta \) | total |
| Number | 6337(30.1%) | 2491(11.9%) | 12172(58.0%) | 21000 |

Figure 1: Plot of the return data of JAL