Estimating and forecasting instantaneous volatility through a duration model: An assessment based on VaR

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Abstract
In order to forecast one-step-ahead volatility, the estimated parameters of a price-change duration model are used to calculate jump intensity. No assumptions are made on any of the possible distributions of log-returns. However, while no distributional assumptions are made, it is practical to choose a suitable distribution e.g., normal, student, etc., including empirical density, when VaR (Value at Risk) with instantaneous volatility is calculated to check prediction performance. The goodness of fit among assumed distributions of log-returns are also compared. The results indicate that fat-tailed distributions, including NIG and Laplace, are well fitted to high frequency data on the Tokyo Stock Exchange (First Section) from 4 January to 28 June 2001.

1 Introduction
During the past decade, the rapid development of computer technology has made high frequency data more readily available. High frequency data provide important information on intraday trades, which, in turn, perform an important role in market microstructure analysis. However, high frequency data have very different characteristics to those found in either daily or weekly data. These include trade intervals (hereafter duration) that are not constant or equidistant, and intraday seasonality (or periodicity). Problematically, while GARCH and SV models are often used to analyze volatility in financial time series, especially with application to derivative pricing and risk management, high frequency data cannot be correctly analyzed using these conventional time series modeling techniques.

In response, Engle and Russell (1998) have proposed a new time series model to deal with high frequency data. We apply their framework to model stochastic time intervals as duration, and derive instantaneous volatility using jump intensity. In order to present empirical verification of the improved forecasting performance of this model, we use the (stochastically occurring) one-step-ahead VaR as calculated from the percentile of an arbitrary distribution fitted to an actual log-return following Giot (2000, 2002). This is also used to compare the goodness of fit to actual log-returns among the assumed distributions.

This paper comprises three sections. In Section 1, we introduce the theoretical background for price duration, a duration model, instantaneous volatility,
seasonal adjustment and the (stochastically occurring) one-step-ahead VaR. In Section 2, we estimate the parameters of a duration model by MLE using the high frequency data on the Tokyo Stock Exchange (First Section), and calculate an instantaneous volatility and (stochastically occurring) one-step ahead VaR. In order to confirm the forecasting performance of these models, likelihood tests are performed. In the final section, and according to the results of the forecasting performance of the (stochastically occurring) one-step-ahead VaR, we compare the goodness of fit among the assumed distributions of log-returns.

2 Theoretical background

2.1 Price duration

Let $S_i$ be a logarithmic stock price, and $r_i = S_i - S_{i-1}$ be its rate of return, where subscript $i$ denotes a $i$th trade, $i = 1, 2, \ldots$. If a cumulative sum of the absolute value of $r_i$, i.e., $|r_i|$, overshoots a predefined constant threshold $c_p$, we regard this as denoting the occurrence of a jump. The time of a jump point is denoted by $T_j$, where $j = 1, 2, \ldots$. Figure 1 shows this procedure. We redefine price duration $d_j = T_j - T_{j-1}$ by $T_j$.

2.2 Duration models

We define $T_j, j = 0, 1, 2, \ldots$ by the time when a jump occurs. $T_j$ is a stochastic process occurring at an irregular time span. The time duration between $j$ and $j-1$ is defined by

$$d_j = T_j - T_{j-1}.$$ 

Sometimes it is assumed that

$$d_j = \psi_j \varepsilon_j \quad \varepsilon_j \sim \text{i.i.d. Non-negative R.V.},$$

where $\psi_j = \mathbf{E}[d_j | \mathcal{F}_{j-1}]$ and $\mathcal{F}_{j-1}$ is information sets generated by the durations until $j-1$. Furthermore, analogous to GARCH, Engle and Russell (1998) advocate the ACD $(p, q)$ (Autoregressive Conditional Duration) model as

$$\psi_j = \omega + \sum_{k=1}^{p} \alpha_k d_{j-k} + \sum_{k=1}^{q} \beta_k \psi_{j-k},$$

where $\omega, \alpha_k, \text{and } \beta_k$ are constants satisfying $\omega > 0, \alpha, \beta \geq 0$. For guaranteeing the stationarity of the model, we impose the following constraint condition on parameters,

$$\sum_{k=1}^{p} \alpha_k + \sum_{k=1}^{q} \beta_k < 1.$$ 

Alternatively, it is assumed that
\[ d_j = \exp(\Psi_j) \epsilon_j \quad \text{or} \quad \epsilon_j = \frac{d_j}{\exp(\Psi_j)} \]

which leads to the Logarithmic-ACD \((p, q)\) model,

\[ \Psi_j = \omega + \sum_{k=1}^{p} \alpha_k \log(d_{j-k}) + \sum_{k=1}^{q} \beta_k \Psi_{j-k}. \tag{1} \]

(See Bauwens and Giot (1999)).

Although there are many possible distributions of \(\epsilon_j\) such as Exponential, Gamma and other non-negative distributions, among others we assume \(\epsilon_j \sim \text{i.i.d. Weibull}\), which is in the wider class than the exponential. We use the logarithmic-ACD model, because there are no restrictions on the sign of the parameters for positivity of \(\exp(\Psi_j)\epsilon_j\). This model is called the Log-Weibull-ACD (LWACD). By definition, the density function of Weibull is

\[ f(d_j) = \frac{\gamma d_j^{\gamma - 1} e^{-\frac{d_j^{\gamma+1}}{\gamma}}}{\Gamma(1 + 1/\gamma)}, \]

where \(\gamma\) denotes the shape parameter and \(\Gamma\) denotes the gamma function. Thus, the log likelihood function of this model is given by

\[ \sum_{j=1}^{N} \left( \log(\gamma) - \log(d_j) + \gamma \log[d_j \Gamma(1 + 1/\gamma)] - \gamma \Psi_j - \left( \frac{d_j \Gamma(1 + 1/\gamma)}{e^{\Psi_j}} \right)^\gamma \right). \]

### 2.3 Instantaneous volatility

First we define a conditional intensity process as

\[ \lambda(T|N(T), T_1, \ldots, T_{N(T)}) = \lim_{\Delta T \to 0} \frac{\Pr(N(T + \Delta T) > N(T)|N(T), T_1, \ldots, T_{N(T)})}{\Delta T}, \]

where \(\Pr(A|B)\) is the usual conditional probability, and \(N(T)\) is the number of events that have occurred by time \(T\). Let the instantaneous volatility be defined as

\[ \hat{\sigma}^2(T) = \lim_{\Delta T \to 0} \frac{1}{\Delta T} \mathbb{E} \left\{ \frac{1}{\Delta T} \left( \frac{S(T + \Delta T) - S(T)}{S(T)} \right)^2 \right\}, \]

where \(S(T)\) is a stock price associated with the arrival time \(T\), and taking limits, we obtain the conditional instantaneous volatility as

\[ \hat{\sigma}^2(T|T_{N(T)}, \ldots, T_1) = \left( \frac{\epsilon_p}{S_{T_{N(T)}}} \right)^2 \lambda(T|T_{N(T)}, \ldots, T_1), \]
where $c_p$ is an arbitrary prescribed constant.

We estimate the intensity by using the Logarithmic-ACD model,

$$
\Psi_j = \omega + \sum_{k=1}^{p} \alpha_k \log(d_{j-k}) + \sum_{k=1}^{q} \beta_k \Psi_{j-k},
$$

with

$$
d_j = \exp(\Psi_j) \epsilon_j \quad \epsilon_j \sim \text{i.i.d. Non-negative R.V.,}
$$

where $\psi_j = E[d_j|\mathcal{F}_{j-1}]$. This is called the Logarithmic-ACD $(p, q)$ model (see Bauwens and Giot (1999)).

As the conditional intensity $\lambda$ of the Logarithmic-ACD model reduces to $1/e^\Psi$, (see Giot (2000)), we obtain the instantaneous volatility of the Logarithmic-ACD version with

$$
\sigma^2(T|\mathcal{F}_{j-1}) = \left(\frac{c_p}{S(T_{j-1})}\right)^2 \frac{1}{e^{\Psi_j}}.
$$

### 2.4 Adjusting intraday seasonality

In general, the duration of high frequency data displays intraday seasonality. For example in the Japanese stock market there is a distinct lunch break, so the plot of its duration is M-shaped. Precise estimation will not result from the use of such data without pretreatment. Therefore, we adjust the seasonality in the following way.

Let $\phi(t_i)$ be a deterministic term and $d_i$ be the non-adjusted duration series $d_i$, and we define a new series $d^*_i$ as

$$
d^*_i = \frac{d_i}{\phi(t_i)},
$$

where $d^*_i$ is called seasonally adjusted duration and $\phi(t_j)$ is called the time of day function. The $\phi(t_j)$ can be estimated by kernel or spline smoothing. Finally we obtain the seasonally adjusted volatility,

$$
(\hat{\sigma}^*)^2(T|\mathcal{F}_{j-1}) = \left(\frac{c_p}{S(T_{j-1})}\right)^2 \frac{1}{e^{\Psi_j}\phi(t_j-1)}.
$$

### 2.5 (Stochastically occurring) one-step-ahead VaR

We can forecast (stochastically occurring) one-step-ahead VaR using the seasonally adjusted instantaneous volatility calculated above. The limit of percentile $1 - \alpha$ is given by

$$
\text{VaR}(j) = z_{1-\alpha} \sqrt{(\hat{\sigma}^*)^2(j|\mathcal{F}_{j-1})} \times \sqrt{M \times 60},
$$
where $M$ denotes a time interval, with $\sqrt{M \times 60}$ changing minutes into seconds in the time interval.

To obtain the percentile $z_{1-\alpha}$, we fit an appropriate distribution to actual log-return data and estimate the parameters of those distributions. The four distributions used are as follows.

**Normal distribution:** The normal distribution is the most basic and commonly employed distribution in investment science, but it is well known that it may not describe the fat tails of most financial time series. The density function is given by

$$f_{\text{Normal}}(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty.$$  

**Student distribution:** The student distribution has fatter tails than the normal distribution. The density function with degree of freedom $m$ is given by

$$f_{\text{Student}}(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\Gamma\left(\frac{m}{2}\right)} \left(1 + \frac{x^2}{m}\right)^{-\frac{m+1}{2}}, \quad -\infty < x < \infty.$$  

If the degree of freedom is larger than 30, then the distribution is almost the same as the normal distribution.

**Laplace (double exponential) distribution:** The density function of Laplace (double exponential) distribution with scale parameter $\beta$ and location parameter $\mu$ is given by

$$f_{\text{Laplace}}(x) = \frac{1}{2\beta} \exp\left\{-\frac{|x - \mu|}{\beta}\right\}, \quad -\infty < x < \infty.$$  

This is obtained by the difference between two independent identical exponential distributions. See Abramowitz and Stegun 1972, p. 930.

**Normal Inverse Gaussian (NIG) distribution:** Following Barndorff-Nielsen (1997), the Normal Inverse Gaussian (NIG) distribution is given by

$$f_{\text{NIG}}(x) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x - \mu}{\delta}\right)^{-1} K_1\left\{\delta q\left(\frac{x - \mu}{\delta}\right)\right\} \exp(\beta x),$$  

where

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \alpha \exp(\delta \sqrt{\alpha^2 - \beta^2} - \beta \mu),$$  

$$q(x) = \sqrt{1 + x^2},$$
and $K_1$ is the modified Bessel function of the third kind with coefficient 1. The parameters $\alpha$, $\beta$, $\mu$ and $\delta$ are satisfied by the condition $0 \leq |\beta| \leq \alpha$, $\mu \in \mathbb{R}$ and $0 < \delta$.

3 Verification of forecasting performance

In this section, we verify the forecasting performance of the (stochastically occurring) one-step-ahead VaR using the framework previously introduced. The data set used comprises high frequency data of the following companies listed on the Tokyo Stock Exchange (First Section) from 4 January to 28 September 2001. These are: NIPPON STEEL (5401), HITACHI (6501), SONY (6578) and TOYOTA (7203). For use later, the data is split into an estimation period (January to March 2001) and a forecast period (April to September 2001).

3.1 Estimation

First we estimate the parameter of the LWACD model from actual data, and calculate the instantaneous volatility. We choose 10 suitable thresholds $c_p$ for each set of data and distribution, and simulate the price duration processes using $c_p$. For the seasonal adjustment, intraday time function $\hat{\phi}$ is estimated by Friedman’s super smoother (see Friedman (1984a, 1984b)). After the adjustment, the LWACD (1, 1) model on the estimation period is given by

$$d_j^* = \exp(\hat{\Psi}_j^*) \epsilon_j, \quad \epsilon_j \sim \text{i.i.d. Weibull}(1, \hat{\gamma}), \quad \hat{\Psi}_j^* = \hat{\omega} + \hat{\alpha} \log(d_{j-1}^*) + \hat{\beta} \hat{\Psi}_{j-1},$$

(7)

We estimate parameters, $\hat{\omega}(c_p)$, $\hat{\alpha}(c_p)$, $\hat{\beta}(c_p)$ and $\hat{\gamma}(c_p)$ by MLE, where the argument $c_p$ in parenthesis indicates that these estimates depend on $c_p$. Then we substitute $\hat{\Psi}_j^*(c_p)$ into (6), allowing us to obtain seasonally adjusted instantaneous volatility $(\hat{\sigma}^*(c_p))^2$. Hereafter for notational convenience the argument $c_p$ is omitted.

3.2 Forecasting

One-step-ahead VaR is calculated by substituting $(\hat{\sigma}^*)^2$ for each $c_p$ into

$$\text{VaR}(j) = z_{1-\alpha} \sqrt{(\hat{\sigma}^*)^2(t|F_{j-1}) + \sqrt{M \times 60}},$$

(8)

thereby enabling us to verify the forecasting performance.

Since the VaR calculated by (8) is derived from the stochastic duration $d_j^*$ in (7), it occurred at a different time from the actual data. Since it is meaningless to discuss forecasting performance by comparing values at different times, we follow Giot and divide the time interval into $M = 10, 15$ or 30, regarding the closing price in each time interval as an observed value for actual data, and
regarding the arithmetic average in each time interval as an observed value for estimated VaR. The reason we use $M = 10, 15$ or $30$ is that if we were to choose a longer time interval, such as one hour, and given the TSE is only open four and a half hours daily, then the sample size would become too small. Likewise, if a shorter time interval is chosen, such as five minutes, then the probability of no trading in any given time interval would become larger.

A percentile $z_{1-\alpha}$ in (8) is calculated by numerical integrations that are executed recursively for each parameter of normal, student, NIG, Laplace and empirical. These parameters are estimated by MLE. Figure 2 shows an example. The solid line represents the actual data, and the dotted line is the VaR calculated by (8), on the forecasting period.

### 3.3 Likelihood Ratio Test

The failure rate of risk is defined as the ratio of the number of times that actual data exceed VaR. Thus if the failure rate of risk is the nearer to the predefined percentile $1 - \alpha$, the forecasting performance of VaR is higher. Table 1 shows an example. We can say that if an entry in Table 1 is the closer to nominal ratios in the head of each column, the VaR has better forecasting performance. The outcome is classified into two cases depending on whether or not the VaR exceeds the actual data. Thus we can regard the outcome as a Bernoulli trial. Then following Kupiec (1995), we can verify the forecasting performance of the VaR.

First, $N$ denotes the number of total trades, and $N_{ex}$ denotes the number that actual data exceed to a limit of VaR within a forecasting period. Let $p$ be the probability of the failure rate of risk, so that $N_{ex}$ follows

$$\Pr\{\nu\} = \binom{N}{\nu} p^\nu (1 - p)^{N-\nu}, \quad \nu = 0, 1, \ldots, N.$$ 

Let us define the likelihood of $N_{ex}$ as

$$L(N_{ex}) = \frac{p_0^{N_{ex}} (1 - p_0)^{N - N_{ex}}}{p^{N_{ex}} (1 - p)^{N - N_{ex}}}.$$ 

By setting $p_0 = 1 - \alpha$ and $\hat{p} = N_{ex}/N$ we can calculate the likelihood, and it is well known that $2 \log L(N_{ex})$ follows $\chi^2(1)$. Hence, we can execute a hypothesis test with the null hypothesis $H_0 : p = p_0$. Table 2 shows an example of log-likelihood ratio statistics $2 \log L(N_{ex})$ for each percentile $1 - \alpha$ of a VaR. If the statistic is not rejected, or accepted, then it is favorable to the assumed distribution. Asterisk “*” denotes the score defined by the following way: when the log-likelihood ratio statistic is not rejected at 90% then 10 points are assigned. Similarly at 75% nine points, ..., 0.1% one point is assigned. Note that the score denotes only an order; the value itself is meaningless. Thus, we can say that when an entry of the table has more *, the density is more suitable for the return.
4 Comparing goodness of fit to distribution

Finally, we compare the goodness of fit for each series using the following steps.

1. Check the normality for actual data.

2. Choose 10 thresholds $c_p$’s for each time interval $M = 10, 15$ or $30$.

3. Execute the likelihood ratio test for VaR processes, five kinds of percentile $1 - \alpha$: $0.050, 0.025, 0.010, 0.005$ and $0.001$, $10$ kinds of thresholds and three kinds of time intervals.

4. Score the results above by regarding the "goodness of forecasting performance of VaR" as the "goodness of fit to a distribution".

5. Execute step 1 to 4 for each actual data and distribution.

6. Rank the goodness of fit.

4.1 Testing normality

Prior to the study of goodness of fit, the normality of the actual data was checked using the Anderson-Darling test. This may detect a fat tail better than the alternate Kolmogorov-Smirnov test. The Anderson-Darling test statistic $A^2$ is defined by

$$A^2 = -N - S,$$

$$S = \frac{1}{N} \sum_{i=1}^{N} (2i - 1) \left[ \log \Phi_i + \log(1 - \Phi_{N+1-i}) \right],$$

where $\Phi$ is the cumulative density function of a standard normal distribution and $N$ is a sample size. The null hypothesis $H_0$: “data follows normal distribution” is rejected at significance level $10\%$, if $A^2 > 0.753$, at significance level $5\%$ $A^2 > 1.159$. Table 3 shows the test result. This table shows that the null hypothesis is rejected for all data sets, i.e., all data do not follow the normal distribution. Furthermore, it also shows that as the time interval becomes larger, the normality becomes greater for all data sets.

4.2 Comparing goodness of fit

Finally we verify the goodness of fit for each distribution. Tables 4, 5 and 6 show the summary of the results of likelihood tests for each of the three time intervals $M = 10, 15$ or $30$, $10$ thresholds and five percentiles of VaR. Table 7 summarizes the results for the time intervals. In this table the top grade is
(the points) $\times$ (the number of time intervals
$\times$ the number of percentile of VaR
$\times$ the number of thresholds).

i.e., $10 \times (3 \times 5 \times 10) = 1500$ is the highest score. Thus, in this table we observe that the NIG distribution is the best fit for each time series, with the exception of TOYOTA. Note, however, that we do not take into consideration empirical density in this summary since it is not a probability distribution.

5 Concluding remarks

In this study, we estimated instantaneous volatility with a duration model. This is used to calculate the (stochastically occurring) one-step-ahead VaR. We also investigated which density provides the best fit to log-return by using the VaR as a measure of fitness. The results indicate that the normal density is not the most suitable for financial time series such as log-returns with fat tails and asymmetry. Finally, we obtained interesting high frequency results for the Tokyo Stock Exchange, which appear plausible and acceptable. There is the possibility that the methods employed in the paper can be used for risk management of downside risk because of the focus on tail behavior.

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References


Table 1: Failure Rate of Risk: SONY (6758) sampled every 15 minutes

<table>
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<tr>
<th></th>
<th>0.050</th>
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<td>0.010059</td>
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<td>0.005007</td>
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Table 2: Likelihood Ratio Test: SONY (6758) sampled every 15 minutes

<table>
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<tr>
<th></th>
<th>0.050</th>
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<th>0.010</th>
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<td>5.0989 *3</td>
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<td>4.5124 *4</td>
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Table 3: The Anderson-Darling test statistics

<table>
<thead>
<tr>
<th></th>
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<th>HITACHI</th>
<th>SONY</th>
<th>TOYOTA</th>
</tr>
</thead>
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<tr>
<td>10 min.</td>
<td>146.7047</td>
<td>20.2217</td>
<td>24.0964</td>
<td>97.7665</td>
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<tr>
<td>15 min.</td>
<td>84.2962</td>
<td>14.5411</td>
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<td>30 min.</td>
<td>29.8662</td>
<td>6.2688</td>
<td>9.577</td>
<td>21.1466</td>
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Table 4: The score summary of likelihood ratio test results [10 min.]

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<th>Laplace</th>
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<td>SONY</td>
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<td>TOYOTA</td>
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Table 5: The score summary of likelihood ratio test results [15 min.]

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<td>SONY</td>
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<td>TOYOTA</td>
<td>237</td>
<td>238</td>
<td>256</td>
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Table 6: The score summary of likelihood ratio test results [30 min.]

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<td>TOYOTA</td>
<td>276</td>
<td>240</td>
<td>258</td>
<td>271</td>
<td>234</td>
<td>Normal</td>
</tr>
</tbody>
</table>

Table 7: The score summary of likelihood ratio test results [10, 15, 30 min.]

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>Student</th>
<th>NIG</th>
<th>Laplace</th>
<th>Empirical</th>
<th>The fittest</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIPPON STEEL</td>
<td>600</td>
<td>605</td>
<td>638</td>
<td>560</td>
<td>616</td>
<td>NIG</td>
</tr>
<tr>
<td>HITACHI</td>
<td>1057</td>
<td>1071</td>
<td>1114</td>
<td>968</td>
<td>1124</td>
<td>NIG</td>
</tr>
<tr>
<td>SONY</td>
<td>799</td>
<td>852</td>
<td>899</td>
<td>800</td>
<td>779</td>
<td>NIG</td>
</tr>
<tr>
<td>TOYOTA</td>
<td>707</td>
<td>684</td>
<td>744</td>
<td>758</td>
<td>657</td>
<td>Laplace</td>
</tr>
</tbody>
</table>
Figure 1: Price duration

Figure 2: SONY (6758) sampled every 15 minutes. (2 April–28 September, 2001)